Int J Theor Phys (2015) 54:4506–4511
DOI 10.1007/s10773-015-2653-8

On the Form of the Optimal Measurement for the Probability of Detection

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Received: 4 November 2014 / Accepted: 8 April 2015 / Published online: 28 April 2015

Abstract We consider the problem of maximizing the probability of detection for an infinite number of mixed states. We show that for linearly independent states there exists a unique simple optimal measurement, generalizing thus a result obtained in finite dimension by Y. Eldar (Phys. Rev. A, **68**, 052303:1-052303:4 2003).

Keywords Probability of detection · Linearly independent states · Mixed states · Simple measurement

1 Introduction

Let $\rho_1, \rho_2, \ldots$ be (finite or infinite number) quantum states (density matrices) on $\mathbb{B}(\mathcal{H})$ — the bounded linear operators on Hilbert space $\mathcal{H}$, of arbitrary dimension, which can occur with some *a priori* probabilities $\pi = (\pi_1, \pi_2, \ldots)$. We want to find, in an optimal way, the state in which the system really is. To this end we perform a *measurement* (called also *strategy*) $\mathbb{M}$, by which is meant a sequence $(M_1, M_2, \ldots)$ of positive operators from $\mathbb{B}(\mathcal{H})$, such that

$$\sum_{i=1}^{\infty} M_i = \mathbb{1}$$

where the series is convergent in the weak operator topology on $\mathbb{B}(\mathcal{H})$. A measurement $\mathbb{M} = (M_1, M_2, \ldots)$ for which all M_i's are pairwise orthogonal projections is called *simple* or *sharp* (see [1]).

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If we receive outcome M_i, we choose state ρ_i. The probability that the true state is ρ_i when measurement will give result M_j is given by $\mathrm{tr}(\rho_i M_j)$. Thus $\mathrm{tr}(\rho_i M_i)$ is the probabilty of guessing correctly state ρ_i. If our guess is ρ_j while the true one is ρ_i, then we pay penalty $L(i, j)$. Function L is called a *loss function.* The *risk function* is defined by the formula

$$R_M(i) = \sum_{j=1}^{\infty} L(i, j)\mathrm{tr}(\rho_i M_j).$$

The expectation of the risk function is called the *Bayes risk,* and denoted by $r(\mathbb{M}, \pi)$, i.e.

$$r(\mathbb{M}, \pi) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \pi_i L(i, j)\mathrm{tr}(\rho_i M_j).$$

Consider the concrete loss function of the form

$$L(i, j) = 1 - \delta_{ij}.$$

Then we have

$$r(\mathbb{M}, \pi) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \pi_i (1 - \delta_{ij})\mathrm{tr}(\rho_i M_j) = 1 - \sum_{i=1}^{\infty} \pi_i \mathrm{tr}(\rho_i M_i).$$

In this case, minimizing Bayes risk is equivalent to maximizing the expression

$$\sum_{i=1}^{\infty} \pi_i \mathrm{tr}(\rho_i M_i).$$

The above expression is the probability of correct guess while performing measurement $\mathbb{M}$, called the *probability of detection*. We shall denote this probability by $\mathbb{P}_D(\mathbb{M})$. We want to find a measurement which maximizes the probability of detection.

The existence of an optimal measurement is discussed in [7] in a general setup. In our case, the following result from [7] is sufficient.

Theorem 1 *There exists a measurement maximizing the probability of detection.*

For two states the solution can be achieved by taking the simple measurement made by the projections on the support of the positive and negative part of the Hermitian operator $\pi_1\rho_1 - \pi_2\rho_2$. Kaniowski [4] did a deeper analysis for two finite dimensional projections with arbitrary a priori probability.

Each state ρ_i has the spectral decomposition

$$\rho_i = \sum_{j=1}^{m_i} \lambda_i^j |\varphi_i^j\rangle\langle\varphi_i^j|, \tag{1}$$

where $\lambda_i^j > 0$ and $m_i \in \{1, 2, \dots, \infty\}$. In our further considerations we assume that the vectors $\{\varphi_n^m\}$ span the Hilbert space $\mathcal{H}$. For arbitrary states it is hard to say anything about the optimal measurement. In the case of finite-dimensional Hilbert space and finite number of a states it is natural to assume linear independence of vectors $\{\varphi_n^m\}$. Then we say that *the states are linearly independent.* For $\dim \mathcal{H} = \infty$ we have a stronger assumption. We say that *states are strongly linearly independent* if vectors $\{\varphi_i^j\}$ are *strongly linearly independent,* i.e. for each i, j we have $\varphi_i^j \notin \overline{\mathrm{Lin}\{\varphi_n^m : n \neq i, m \neq j\}}$.

A state ρ is called *pure* if it has the form $|\varphi\rangle\langle\varphi|$ for some unit vector $\varphi \in \mathcal{H}$, otherwise a state is called *mixed.* For $\dim \mathcal{H} < \infty$ and pure states Kennedy [5, 6] obtained the following result.

Theorem (**Kennedy** [5, 6]**, 1973,74**) *Let pure states $\rho_1, \rho_2, \ldots, \rho_n$ be linearly independent. Then there exists a unique measurement maximizing the probability of detection and this measurement is simple.*

It turns out that this result holds also for $\dim \mathcal{H} = \infty$.

Theorem (**Łuczak** [7]**, 2009**) *Let pure states $\rho_1, \rho_2, \ldots$ be strongly linearly independent. Then there exists a unique measurement maximizing the probability of detection and this measurement is simple.*

For $\dim \mathcal{H} < \infty$ and arbitrary states Eldar [2] obtained the following result.

Theorem (**Eldar** [2]**, 2003**) *Let the states $\rho_1, \rho_2, \ldots, \rho_n$ be linearly independent. Then there exists a unique measurement maximizing the probability of detection and this measurement is simple.*

A natural question is whether the Eldar result can be generalized to infinite dimension. In this paper we show that the answer is positive.

2 Optimal Measurement

From Theorem 1 there exists a measurement which maximize the probability of detection. One of the most useful optimal measurement conditions was obtained in [3, Theorem II.2.2] and says that

Theorem (**Holevo condition**) *Let $\mathbb{M} = (M_1, M_2, \ldots)$ be an optimal measurement for the probability of detection. Then the operator $\Lambda = \sum_i \pi_i \rho_i M_i$ is Hermitian and*

$$(\Lambda - \pi_i \rho_i) M_i = 0 \qquad \textit{for all } i's.$$

Let $\rho_1, \rho_2, \ldots$ be states of the the form (1). Assume that for some k we have

$$\varphi_k^j \notin \overline{\mathrm{Lin}\{\varphi_n^m : n \neq k, m \neq j\}}, \, j = 1, 2, \ldots, m_k. \tag{2}$$

Lemma 1 *If $\mathbb{M} = (M_1, M_2, \ldots)$ is an optimal measurement, then for each j's*

$$M_k \varphi_k^j \neq 0.$$

Proof We use the method from the proof of Lemma 4 in [7]. Assume that we have e.g., $M_1 \varphi_1^1 = 0$. Let Q be a projection onto

$$\overline{\mathrm{Lin}\left(\left\{\varphi_i^j : j = 1, 2, \ldots, m_i, i = 1, 2, \ldots\right\} \setminus \{\varphi_1^1\}\right)}.$$

Define a new measurement $\hat{\mathbb{M}} = (\mathbb{1} - Q + QM_1Q, QM_2Q, QM_3Q, \ldots)$. Since $Q\rho_1Q = \sum_{k=2}^{m_1} \lambda_1^k |\varphi_1^k\rangle\langle\varphi_1^k|$ and $Q\rho_iQ = \rho_i, i = 2, 3, \ldots,$ we have

$$\begin{aligned}\mathbb{P}_D(\hat{\mathbb{M}}) &= \pi_1 \mathrm{tr}(\rho_1(\mathbb{1} - Q)) + \pi_1 \mathrm{tr}(\rho_1 QM_1Q) + \sum_{i=2}^{\infty} \pi_i \mathrm{tr}(\rho_i QM_iQ) \\ &= \pi_1 \mathrm{tr}(\rho_1(\mathbb{1} - Q)) + \pi_1 \mathrm{tr}\left(\sum_{k=2}^{m_1} \lambda_1^k |\varphi_1^k\rangle\langle\varphi_1^k| M_1\right) + \sum_{i=2}^{\infty} \pi_i \mathrm{tr}(\rho_i M_i) \\ &= \pi_1 \mathrm{tr}(\rho_1(\mathbb{1} - Q)) + \mathbb{P}_D(\mathbb{M}).\end{aligned}$$

From the above and the optimality of the measurement $\mathbb{M}$ we obtain that $\mathrm{tr}(\rho_1(\mathbb{1}-Q)) = 0$. Therefore $\mathrm{tr}(\rho_1 Q) = 1$. This gives

$$\sum_{j=1}^{m_1} \lambda_1^j \left\langle \varphi_1^j | Q\varphi_1^j \right\rangle = 1 \Leftrightarrow \forall_{j=1,2,\ldots,m_1} \left\langle \varphi_1^j | Q\varphi_1^j \right\rangle = 1 \Leftrightarrow \forall_{j=1,2,\ldots,m_1} Q\varphi_1^j = \varphi_1^j.$$

This contradicts the relation

$$\varphi_1^1 \notin \overline{\mathrm{Lin}\left(\{\varphi_i^j : j = 1, 2, \ldots, m_i, i = 1, 2, \ldots\} \setminus \left\{\varphi_1^1\right\}\right)}.$$

□

Before the main theorem we show an interesting result.

Theorem 2 *If* $\mathbb{M} = (M_1, M_2, \ldots)$ *is an optimal measurement, then* M_k *is a nonzero uniquely determined projection.*

Proof From the Holevo condition we have

$$\sum_{i \neq k} \pi_i \rho_i M_i M_k = \pi_k \rho_k \left(M_k - M_k^2\right).$$

Therefore for all $\xi \in \mathcal{H}$ we obtain

$$\sum_{i \neq k} \sum_j \pi_i \lambda_i^j \left\langle \varphi_i^j | M_i M_k \xi \right\rangle \varphi_i^j = \sum_j \pi_k \lambda_k^j \left\langle \varphi_k^j | \left(M_k - M_k^2\right) \xi \right\rangle \varphi_k^j.$$

From the above and assumption (2) on vectors $\left\{\varphi_i^k\right\}$ we have for all $\xi \in \mathcal{H}$ and $j's$

$$\left\langle \varphi_k^j | \left(M_k - M_k^2\right) \xi \right\rangle = 0,$$

so

$$M_k^2 \varphi_k^j = M_k \varphi_k^j.$$

Hence and from Lemma 1 there exists j such that $M_k \varphi_i^j$ is an eigenvector of the operator M_i with eigenvalue equal to 1. Let J_k be the set of all such $j's$.

Define a sequence $\hat{\mathbb{M}} = (\hat{M}_1, \hat{M}_2, \ldots)$ where $\hat{M}_i = M_i, i \neq k$ and $\hat{M}_k$ is a nonzero projection onto

$$\overline{\mathrm{Lin}\left\{M_k \varphi_k^j : j \in J_k\right\}}.$$

Because $\hat{M}_k \leq M_k$ we have $\mathrm{tr}(\rho_k \hat{M}_k) \leq \mathrm{tr}(\rho_k M_k)$. On the other hand, since $\hat{M}_k$ commutes with M_k

$$\mathrm{tr}(\rho_k M_k) = \mathrm{tr}(\hat{M}_k M_k \rho_k) = \mathrm{tr}\left(\rho^{\frac{1}{2}} \hat{M}_k M_k \hat{M}_k \rho_k^{\frac{1}{2}}\right) \leq \mathrm{tr}\left(\rho^{\frac{1}{2}} \hat{M}_k \rho_k^{\frac{1}{2}}\right) = \mathrm{tr}(\rho_k \hat{M}_k).$$

These inequalities give $\mathrm{tr}(\rho_k M_k) = \mathrm{tr}(\rho_k \hat{M}_k)$, hence

$$\mathbb{P}_D(\hat{\mathbb{M}}) = \mathbb{P}_D(\mathbb{M}). \tag{3}$$

Assume that $T = \mathbb{1} - \sum_{i=1}^{\infty} \hat{M}_i$ is a nonzero operator. Then there exist numbers i, j such that $\langle \varphi_i^j | T \varphi_i^j \rangle > 0$ because vectors $\{\varphi_n^m\}$ span the space $\mathcal{H}$. Define a new measurement $\hat{\mathbb{N}} = (\hat{M}_1, \hat{M}_2, \ldots, \hat{M}_i + T, \ldots)$. From (3) we have

$$\mathbb{P}_D(\hat{\mathbb{N}}) = \sum_{n=1}^{\infty} \pi_n \mathrm{tr}(\rho_n \hat{M}_n) + \pi_i \mathrm{tr}(\rho_i T) = \mathbb{P}_D(\hat{\mathbb{M}}) + \pi_i \mathrm{tr}(\rho_i T) > \mathbb{P}_D(\mathbb{M}),$$

which is impossible. Therefore $T = 0$ and $\sum_{i=1}^{\infty} \hat{\mathbb{M}}_i = \mathbb{1}$. Hence $M_k = \hat{M}_k$.

Assume that $\mathbb{M} = (M_1, M_2, \ldots)$ and $\mathbb{N} = (N_1, N_2, \ldots)$ are two distinct optimal measurements such that $M_k \neq N_k$. From the above M_k and N_k are projections. Of course $\frac{1}{2}\mathbb{M} + \frac{1}{2}\mathbb{N}$ is also an optimal measurement and $\frac{1}{2}M_k + \frac{1}{2}N_k$ is a projection. Then

$$\begin{aligned} \frac{1}{2}M_k + \frac{1}{2}N_k = \left(\frac{1}{2}M_k + \frac{1}{2}N_k\right)^2 &\Leftrightarrow \\ 2M_k + 2N_k = M_k + M_k N_k + N_k M_k + N_k &\Leftrightarrow \\ (M_k - N_k)^2 = 0 \Leftrightarrow M_k = N_k&, \end{aligned}$$

a contradiction. Consequently, M_k is uniquely determined. □

Let the states $\rho_1, \rho_2, \ldots$ of the form (1) be strongly linearly independent. Our main theorem is

Theorem 3 *There exists a unique measurement maximizing the probability of detection and this measurement is simple with the nonzero outcomes.*

Proof Let $\mathbb{M} = (M_1, M_2, \ldots)$ be a measurement maximizing the probability of detection. From Theorem 2 each M_i is a nonzero uniquely determined projection. Of course all M_i are mutually orthogonal because

$$M_1 + M_2 + \ldots = \mathbb{1}.$$

Therefore $\mathbb{M}$ is the unique simple measurement with the nonzero outcomes.

□

Suppose now that $\rho_1, \rho_2, \ldots$ are arbitrary states linearly independent or not. The next theorem shows a relation between the ranges of elements of an optimal measurement and the ranges of the states in question.

Theorem 4 *Let $\mathbb{M} = (M_1, M_2, \ldots)$ be a measurement maximizing the probabilty of detection. Then*

$$dimRange M_i \leq dimRange \rho_i$$

for all i's.

Proof Let Λ be the operator in the Holevo condition. From this condition we have

$$(\Lambda - \pi_i \rho_i) M_i = 0 \qquad \text{for all } i's. \tag{4}$$

Let ξ be an arbitrary nonzero vector in $\mathcal{H}$. Then exists k, j such that

$$\langle \xi | \varphi_k^j \rangle \neq 0, \tag{5}$$

because vectors $\{\varphi_n^m\}$ span the space $\mathcal{H}$. From [3, Theorem II.2.2] we obtain

$$\Lambda \geq \pi_i \rho_i \qquad \text{for all } i's. \tag{6}$$

Therefore

$$\langle \xi | \Lambda \xi \rangle \geq_{(6)} \langle \xi | \pi_k \rho_k \xi \rangle \geq \pi_k \left\langle \xi | \varphi_k^j \right\rangle \left\langle \varphi_k^j | \xi \right\rangle >_{(5)} 0,$$

so the operator Λ is invertible.

Consequently, condition (4) is of the form

$$M_i = \pi_i \Lambda^{-1} \rho_i M_i.$$

This implies

$$\dim\mathrm{Range} M_i \leq \dim\mathrm{Range} \rho_i.$$

□

As a corollary we obtain the main result of [7].

Corollary 1 *Let pure states $\rho_1, \rho_2, \ldots$ be strongly linearly independent. Then there exists a unique measurement maximizing the probability of detection and this measurement is simple. The outcomes of this measurement are rank one operators.*

Proof The first part of the corollary is a consequence of Theorem 3. From Theorem 4 the outcomes of the optimal measurement are zero or rank one operators but Lemma 1 implies that the outcomes can't be zero operators. □

Acknowledgments The author would like to thank A. Łuczak for valuable comments which helped to improve the final version of the paper.

References

1. Busch, P., Grabowski, M., Lahti, P.J.: Operational Quantum Physics. Springer-Verlag, Berlin (1995)
2. Eldar, Y.C.: Von Neumann measurement is optimal for detecting linearly independent mixed quantum state. Phys. Rev. A **68**, 052303:1–052303:4 (2003)
3. Holevo, A.S.: Investigations in the general theory of statistical decisions. In: Proceedings of the Steklov Institute of Mathematics, vol. 124, issue 3 (1978)
4. Kaniowski, K.: Optimal ambiguous discrimination between the states given by two projections. Math. Slovaca **64**, 247–258 (2014)
5. Kennedy, R.S.: On the optimum receiver for the M-ary linearly independent pure state problem . M.I.T Res. Lab. Electron. Quart. Prog. Rep **110**, 142 (1973)
6. Kennedy, R.S.: Uniqueness of the optimum receiver for the M-ary pure state problem. M.I.T. Res. Lab. Electron. Quart. Prog. Rep **113**, 129 (1974)
7. Łuczak, A.: Maximizing the probability of detection for pure states. J. Math. Phys. **50**, 053502:1–053502:14 (2009)